Recurrent system of equation (**Variant of original version**)

Find all real solutions of the following system of equations

$x_kx_{k+1}+1=4x_k, k=1,2,\dots,2019$

$x_{2020}x_1+1=4x_1$

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We will solve the following general version of the problem:

For any natural $m\geq 2$ find all real solutions of the following system of equations

(**1**) $\begin{cases} x_kx_{k+1}+1=4x_k, k=1,2,\dots,m-1 \\ x_mx_1+1=4x_1 \end{cases}$.

Let $t_k:=4-x_k, k=1,2,\dots,n$. Then (**1**)$\Leftrightarrow \begin{cases} x_{k+1}=\dfrac{1}{4-x_k}, k=1,2,\dots,m-1 \\ x_1=\dfrac{1}{4-x_m} \end{cases} \Leftrightarrow$

(**2**) $\begin{cases} t_{k+1}=4-\dfrac{1}{t_k}, k=1,2,\dots,m-1 \\ t_1=4-\dfrac{1}{t_m} \end{cases}$ and $x_k=4-t_k, k=1,2,\dots,m.$

Consider a sequence $(t_n)_{n\in\mathbb{N}}$ defined by $t_1=t\neq 0, t_{n+1}=4-\dfrac{1}{t_n}, n\in\mathbb{N}.$

Let $p_n:=t_1t_2\dots t_n, n\in\mathbb{N}$ and $p_0=1.$ Then $p_1=t,\ t_n:=\dfrac{p_n}{p_{n-1}}, n=1,2,\dots,$ and $t_{n+1}=4-\dfrac{1}{t_n}$ becomes $\dfrac{p_{n+1}}{p_n}=4-\dfrac{p_{n-1}}{p_n}\Leftrightarrow p_{n+1}-4p_n+p_{n-1}=0, n\in\mathbb{N}.$

Consider sequence (a_n) defined by $a_0=0, a_1=1$ and $a_{n+1}-4a_n+a_{n-1}=0, n\in\mathbb{N}.$

Since $\det\begin{pmatrix} a_{n+1} & a_n \\ a_n & a_{n-1}\end{pmatrix}\neq 0, n\in\mathbb{N}$ ($a_{n+2}a_n-a_{n+1}^2=a_{n+1}a_{n-1}-a_n^2, n\in\mathbb{N}$ implies $a_{n+1}a_{n-1}-a_n^2=a_2a_0-a_1^2=-1$) then p_n can be represent as linear combination of a_n and a_{n+1}, namely $p_n=c_1a_n+c_2a_{n+1}, n\in\mathbb{N}\cup\{0\}$ for some $c_1,c_2\in\mathbb{R}.$

Using initial conditions $p_0=1, p_1=t$ we obtain $\begin{cases} 1=c_1a_0+c_2a_1 \\ t=c_1a_1+c_2a_2\end{cases}\Leftrightarrow\begin{cases} c_2=1 \\ c_1=t-4\end{cases}$.

Hence, $p_n=(t-4)a_n+a_{n+1}$ and $t_n=\dfrac{(t-4)a_n+a_{n+1}}{(t-4)a_{n-1}+a_n}, n\in\mathbb{N}.$

Coming back to the system (**2**) and noting that* $a_n>0$ for any $n\in\mathbb{N}$ we obtain

$t_1=4-\dfrac{1}{t_m}\Leftrightarrow t=\dfrac{(t-4)a_m+a_{m+1}}{(t-4)a_{m-1}+a_m}\Leftrightarrow (t-4)a_m+a_{m+1}=t(t-4)a_{m-1}+ta_m\Leftrightarrow$

$a_{m-1}t^2-4a_{m-1}t+(4a_m-a_{m+1})=0\Leftrightarrow a_{m-1}t^2-4a_{m-1}t+a_{m-1}=0, m\geq 2\Leftrightarrow$

$t^2-4t+1=0\Leftrightarrow t=2\pm\sqrt{3}\Leftrightarrow x=4-\left(2\pm\sqrt{3}\right)=2\mp\sqrt{3}.$

Thus, $x_1=x_2=\ldots=x_m=2+\sqrt{3}$ and $x_1=x_2=\ldots=x_m=2-\sqrt{3}$ are all real solution of the system (**1**).

* Since $a_1=1, a_2=4$ then for any $n\in\mathbb{N}$ assuming $a_{n-1}<a_n$ and $a_n>0$ we obtain $a_{n+1}-a_n=2a_n+(a_n-a_{n-1})\Rightarrow a_{n+1}-a_n>a_n-a_{n-1}\Rightarrow a_{n+1}-a_n>0\Leftrightarrow a_{n+1}>a_n$ and, therefore, $a_{n+1}>0.$ Thus, by Math Induction $a_n>0, \forall n\in\mathbb{N}.$